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Quantum gravidynamics II. Path integrals with spin

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Abstract. Spin operators in a path integral can be collected into a path operator, which appears quite separate from the action and outside the exponential. In this way the problem of non-commuting operators inside the exponential is avoided.

In curved space–time spinor matrices in the path operator must be referred to different frames at different points along the path. The components of the spinor matrices referred to different frames must be connected by means of the spinor parallel propagator.

1. Introduction

In part I (Clutton-Brock 1975) we discussed path integrals without spin. We must now introduce spin, and discuss path integrals for Dirac particles.

If the path integrals are taken over coordinate space only, spin operators will appear in the action S. The presence of non-commuting spin operators inside the exponential $\exp(iS)$ complicates the path integral. Hoyle and Narlikar (1971) show that, in their action-at-a-distance electrodynamics, the action turns out to be a scalar in spite of containing a product of γ matrices. One will not always be so fortunate, but Hamilton and Schulman (1971) have shown how a product integral, which is a generalization of a path integral, is capable of handling exponentials of products of non-commuting operators.

We shall take path integrals over both coordinate and momentum space. The spin operators then appear outside the exponential, quite separate from the action. As in the Klein–Gordon case of part I, the transition amplitude can be written in the form

$$T_{\mathrm{FI}} = \int \bar{\psi}_{\mathrm{F}} Q(\text{path}) \psi_{\mathrm{I}} \exp[iS(V_{\mathrm{F}} \leftarrow \text{path} \leftarrow V_{\mathrm{I}})] \, d(\text{path}), \tag{1.1}$$

where S is the classical action and Q is the path operator. In the Dirac case, the classical action is entirely independent of the spin, and the spin operators are all collected into the path operator which appears outside the exponential. The purpose of this part II is to find the path operator for a Dirac particle.

In curved space–time spin does introduce an extra complication. Spinor components must be referred to a tetrad reference frame, which in curved space–time cannot be constant. The spinors along a path must therefore be connected by the spinor parallel propagator. The spinor parallel propagator is the two-point spinor analogous to the tensor parallel propagator. The general theory of two-point spinors has been discussed by Lichnerowicz (1964); for our purpose we need only a few simple properties of the

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spinor parallel propagator which can be obtained from the Lorentz invariance of the Dirac equation.

The spinor parallel propagator can be used to define covariant differentiation of spinors, which appears in the curved-space version of Dirac's equation. The exact kernel is the Green function of Dirac's equation; an approximate kernel can be found, as in part I, by a suitable generalization of the free-particle flat-space kernel. This approximation is accurate for small intervals, and a path integral is a convolution of kernels for many small intervals. We can find the path operator for a Dirac particle simply by collecting up the appropriate terms in the resulting expression for the path integral.

2. The spinor parallel propagator

Just as the tensor parallel propagator $g^k(x, x')_m$ takes a tensor T by parallel transport from x' to x,

$$T^k(x) = g^k(x, x')_m T^m(x'), \tag{2.1}$$

so the spinor parallel propagator $\Lambda(x, x')$ takes a spinor ψ by parallel transport from x' to x:

$$\psi(x) = \Lambda(x, x')\psi(x'). \tag{2.2}$$

The spinor frame is connected to the tensor frame via a tetrad field in such a way that the tetrad components γ^α of Dirac's gamma matrices are constant. From the invariance of the Dirac equation

$$(\gamma^\alpha q_\alpha + m)\psi = 0, \tag{2.3}$$

we can derive the commutation relation

$$\Lambda(x, x')\gamma^\alpha = g^\alpha(x, x')_\beta \gamma^\beta \Lambda(x', x), \tag{2.4}$$

which can be put in the equivalent forms

$$\gamma^\alpha = g^\alpha(x, x')_\beta \Lambda(x, x')\gamma^\beta \Lambda(x', x) \tag{2.5}$$

and

$$\gamma^\alpha g_\alpha(x, x')^\beta = \Lambda(x, x')\gamma^\beta \Lambda(x', x). \tag{2.6}$$

If we express the parallel propagators in terms of elementary generators P and Π,

$$g^\alpha(x, x')_\beta = \exp[P^\alpha(x, x')_\beta] = \eta^\alpha_\beta + P^\alpha_\beta + \tfrac{1}{2}P^\alpha_\mu P^\mu_\beta + \ldots, \tag{2.7}$$

$$\Lambda(x, x') = \exp[\Pi(x, x')] = I + \Pi + \tfrac{1}{2}\Pi^2 + \ldots, \tag{2.8}$$

then we can obtain from the commutation relation (2.6) the relationship

$$\Pi = -\tfrac{1}{4}\sigma^{\alpha\beta}P_{\alpha\beta} \qquad \text{with} \qquad \sigma^{\alpha\beta} = \tfrac{1}{2}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha). \tag{2.9}$$

for signature $(+\,+\,+\,-)$. For the opposite signature the sign is reversed.

We can express the covariant derivative of both tensors and spinors in terms of the parallel propagators for an infinitesimal interval

$$D_m T^k(x)\,\mathrm{d}x^m = g^k(x, x+\mathrm{d}x)_m T^m(x+\mathrm{d}x) - T^k(x), \tag{2.10}$$

$$D_m \psi(x)\,\mathrm{d}x^m = \Lambda(x, x+\mathrm{d}x)\psi(x+\mathrm{d}x) - \psi(x). \tag{2.11}$$

Using (2.7)–(2.9) we find

$$\Lambda(x, x+\mathrm{d}x) = I - \tfrac{1}{4}\sigma^{\alpha\beta}[g_\alpha(x, x+\mathrm{d}x)_\beta - \eta_{\alpha\beta}]. \tag{2.12}$$

Applying (2.10) to the tetrad $\lambda^k_\beta(x)$ we find

$$\lambda^k_{\beta|m}\,\mathrm{d}x^m = g^k(x, x+\mathrm{d}x)_m \lambda^m_\beta(x+\mathrm{d}x) - \lambda^k_\beta(x), \tag{2.13}$$

or

$$\lambda_{\alpha k}\lambda^k_{\beta|m}\,\mathrm{d}x^m = g_\alpha(x, x+\mathrm{d}x)_\beta - \eta_{\alpha\beta}, \tag{2.14}$$

so

$$\Lambda(x, x+\mathrm{d}x) = I - \tfrac{1}{4}\sigma^{\alpha\beta}\lambda_{\alpha k}\lambda^k_{\beta|m}\,\mathrm{d}x^m. \tag{2.15}$$

The last term on the right is the *spinorial connection*

$$\Gamma_m = -\tfrac{1}{4}\sigma^{\alpha\beta}\lambda_{\alpha k}\lambda^k_{\beta|m}, \tag{2.16}$$

in terms of which the covariant derivative of a spinor is

$$D_m\psi = (\partial_m + \Gamma_m)\psi. \tag{2.17}$$

The covariant derivative of the adjoint spinor is

$$\bar{\psi}\overleftarrow{D}_m = \bar{\psi}(\overleftarrow{\partial}_m - \Gamma_m), \tag{2.18}$$

which follows from the invariance of $\bar{\psi}\psi$ or alternatively from the adjoint of (2.11) together with

$$\bar{\Lambda} = \gamma^0\tilde{\Lambda}\gamma^0 = \Lambda^{-1}. \tag{2.19}$$

Dirac's equation in curved space–time is

$$(-\mathrm{i}\gamma^k D_k + m)\psi = 0, \tag{2.20}$$

which agrees with the form obtained by Fock (1929) and Dirac (1958).

3. An approximate kernel for Dirac particles

For Dirac particles, the relationship between the wavefunction at a point x'' and the wavefunction on a three-surface V' enclosing x'' is

$$\psi(x'') = \int K(x'', x')\mathcal{N}'\psi(x')\,\mathrm{d}V', \tag{3.1}$$

where,

$$\mathcal{N} = \gamma^k N_k \tag{3.2}$$

and N_k is a vector orthogonal to the invariant element of three-surface $\mathrm{d}V$ of magnitude such that

$$N_k\,\mathrm{d}V = \sqrt{-g}\,\epsilon_{klmn}\,\mathrm{d}x^l\,\mathrm{d}x^m\,\mathrm{d}x^n \tag{3.3}$$

and $K(x'', x')$ is the spinor kernel. As in part I we can use Gauss's theorem to transform (3.1) into an integral over the four-content C' enclosed by V':

$$\psi(x'') = \int D'_k\{K(x'', x')\gamma^k\psi(x')\}\,\mathrm{d}C'. \tag{3.4}$$

The product $K\gamma^k\psi$ must transform like a spinor at x'', but a tensor (not a spinor) at x', so that $K(x'', x')$ transforms like an adjoint spinor at x'. In the appendix we show that the tensor γ^k has zero divergence

$$D_k\gamma^k = 0, \tag{3.5}$$

so that, if K is an adjoint spinor,

$$D_k(K\gamma^k\psi) = K(\overleftarrow{D}_k\gamma^k + \gamma^k\overrightarrow{D}_k)\psi. \tag{3.6}$$

Now if we substitute for $D_k\psi$ using Dirac's equation we obtain

$$D'_k\{K(x'', x')\gamma^k\psi(x')\} = K(x'', x')(\overleftarrow{D}'_k\gamma^k - \mathrm{i}m)\psi(x'). \tag{3.7}$$

Putting (3.7) into (3.4) tells us immediately that the differential equation satisfied by the kernel is

$$K(x'', x')(\overleftarrow{D}'_k\gamma^k - \mathrm{i}m) = \mathscr{I}(x'', x'), \tag{3.8}$$

where $\mathscr{I}(x'', x')$ is now the spinorial identity kernel:

$$\mathscr{I}(x'', x') = (g''g')^{-1/4}\delta^4(x'' - x')I. \tag{3.9}$$

In flat space–time equation (3.8) has the solution

$$K(x'', x') = \frac{\mathrm{i}}{(2\pi)^4}\int \exp[\mathrm{i}p_k(x'' - x')^k]\frac{-\gamma^l p_l + m}{\eta^{ab}p_a p_b + m^2}\,\mathrm{d}^4p, \tag{3.10}$$

where the integral over p_0 is taken along the Feynman contour so as to include the correct sheet of the momentum surface. As in part I we can generalize this to obtain an approximate kernel in curved space–time. We work in terms of the mechanical momentum q which is defined so as to give the momentum surface the simple form

$$\Omega(q) \equiv \eta^{\alpha\beta}q_\alpha q_\beta + m^2 = 0. \tag{3.11}$$

The 'element of momentum surface' is

$$\mathrm{d}\Omega(q) = \frac{1}{(2\pi)^4}\frac{\mathrm{d}^4q}{\eta^{\alpha\beta}q_\alpha q_\beta + m^2}, \tag{3.12}$$

where the correct sheet is assumed according to the Feynman prescription. In addition we use the notation $w(q)$ to denote the 'Dirac square root' of $\Omega(q)$, or

$$w(q) = -\gamma^\alpha q_\alpha + m. \tag{3.13}$$

We must also replace $p_k(x'' - x')^k$ by the invariant integral

$$p_k(x'' - x')^k \rightarrow S\begin{pmatrix} x'' \leftarrow x' \\ q \end{pmatrix} = \int_{x'}^{x''} p_k(q, x)\,\mathrm{d}x^k. \tag{3.14}$$

These prescriptions, which were developed in part I for the covariant Klein–Gordon kernel, would lead to the Dirac kernel

$$\hat{K}(x'', x') \rightarrow \mathrm{i}\int \exp\left[\mathrm{i}S\begin{pmatrix} x'' \leftarrow x' \\ q \end{pmatrix}\right]w(q)\,\mathrm{d}\Omega(q), \tag{3.15}$$

but this is not yet covariant. For the spinor matrix $w(q)$ of (3.13) must be referred to some point x_q on the geodesic joining x' and x'', but the kernel $\hat{K}(x'', x')$ has to connect spinors at x' with spinors at x''. This involves the transport of a spinor from x' to x_q, multiplication by $w(q)$, and transport from x_q to x''. The transport can be effected by the spinor parallel propagator, which gives the substitution

$$w(q) \to \Lambda(x'', x_q)w(q)\Lambda(x_q, x'). \tag{3.16}$$

This replacement is independent of where on the geodesic x_q lies, since for example

$$w(q'')\Lambda(x'', x') = \Lambda(x'', x')w(q'), \tag{3.17}$$

as may be verified using the commutation relation (2.4). The covariant kernel for a Dirac particle is therefore

$$\hat{K}(x'', x') = \mathrm{i}\Lambda(x'', x_q) \int \exp\left[\mathrm{i}S\begin{pmatrix} x'' \leftarrow x' \\ q \end{pmatrix}\right] w(q)\, \mathrm{d}\Omega(q)\Lambda(x_q, x'). \tag{3.18}$$

This is an approximation to the exact kernel which is accurate when the interval $x'' \leftarrow x'$ is small.

4. The path operator for Dirac particles

The transition amplitude for a Dirac particle to go from a state ψ_I defined on a three-surface V_I to a state ψ_F on a three-surface V_F enclosed by V_I is

$$T_\mathrm{FI} = \int\int \bar{\psi}_\mathrm{F}\mathcal{N}_\mathrm{F}K(x_\mathrm{F}, x_\mathrm{I})\mathcal{N}_\mathrm{I}\psi_\mathrm{I}\, \mathrm{d}V_\mathrm{F}\, \mathrm{d}V_\mathrm{I}. \tag{4.1}$$

To see how to express this as a path integral, consider as in part I the simple path

$$\{\text{path}\} = \begin{Bmatrix} x_1 \leftarrow x_3 \leftarrow x_5 \\ q_2 \leftarrow q_4 \end{Bmatrix}. \tag{4.2}$$

The kernel $K(x_1, x_5)$ is the convolution of two kernels $K(x_1, x_3)$ and $K(x_3, x_5)$, and if we use the form (3.18) for these kernels, we obtain

$$\begin{aligned} K(x_1, x_5) &= \int K(x_1, x_3)\mathcal{N}_3K(x_3, x_5)\, \mathrm{d}V_3 \\ &= \mathrm{i}^2 \int\int\int \Lambda_{1,2}w_2\Lambda_{2,3}\mathcal{N}_3\Lambda_{3,4}w_4\Lambda_{4,5} \\ &\qquad \times \exp\left[\mathrm{i}S\begin{pmatrix} x_1 \leftarrow x_3 \leftarrow x_5 \\ q_2 \leftarrow q_4 \end{pmatrix}\right] \mathrm{d}\Omega_2\, \mathrm{d}V_3\, \mathrm{d}\Omega_4. \end{aligned} \tag{4.3}$$

We have used here the abbreviated notation

$$\Lambda_{1,2} = \Lambda(x_1, x_2), \qquad w_2 = w(q_2), \qquad \mathrm{d}\Omega_2 = \mathrm{d}\Omega(q_2). \tag{4.4}$$

If we identify x_1 with x_F and x_5 with x_I and substitute (4.4) into (4.1), we obtain

$$\begin{aligned} T_\mathrm{FI} = \mathrm{i}^2 \int \cdots \int & \bar{\psi}_\mathrm{F}(x_1)\mathcal{N}_1\Lambda_{1,2}w_2\Lambda_{2,3}\mathcal{N}_3\Lambda_{4,5}\mathcal{N}_5\psi_\mathrm{I}(x_5) \\ & \times \exp\left[\mathrm{i}S\begin{pmatrix} x_1 \leftarrow x_3 \leftarrow x_5 \\ q_2 \leftarrow q_4 \end{pmatrix}\right] \mathrm{d}V_1\, \mathrm{d}\Omega_2\, \mathrm{d}V_3\, \mathrm{d}\Omega_4\, \mathrm{d}V_5. \end{aligned} \tag{4.5}$$

This has the path integral form

$$T_{\mathrm{FI}} = \int \bar{\psi}_{\mathrm{F}} Q(\text{path}) \psi_{\mathrm{I}} \exp[\mathrm{i}S(V_{\mathrm{F}} \leftarrow \text{path} \leftarrow V_{\mathrm{I}})] \, \mathrm{d}(\text{path}), \tag{4.6}$$

provided we identify

$$\mathrm{d}(\text{path}) = \mathrm{d}V_1 \, \mathrm{d}\Omega_2 \, \mathrm{d}V_3 \times \ldots \times \mathrm{d}V_{2N-1} \, \mathrm{d}\Omega_{2N} \, \mathrm{d}V_{2N+1} \tag{4.7}$$

as the 'element of path' for an N-step path, and

$$Q(\text{path}) = \mathrm{i}^N \mathcal{N}_1 \Lambda_{1,2} w_2 \Lambda_{2,3} \mathcal{N}_3 \times \cdots \times \mathcal{N}_{2N-1} \Lambda_{2N-1,2N} w_{2N} \Lambda_{2N,2N+1} \mathcal{N}_{2N+1} \tag{4.8}$$

as the path operator for a Dirac particle. Notice that the action inside the exponential is the classical action

$$S(V_{\mathrm{F}} \leftarrow \text{path} \leftarrow V_{\mathrm{I}}) = \sum_{n=1}^{N} \int_{x_{2n-1}}^{x_{2n+1}} p_k(q_{2n}, x) \, \mathrm{d}x^k \tag{4.9}$$

which does not involve the spin; spin and the spinor operators are confined to the path operator (4.8) which is outside the exponential. The problem of non-commuting operators inside the exponential is entirely avoided.

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Appendix

We now prove that the divergence of the tensor γ^k is zero:

$$D_k \gamma^k = 0, \tag{A.1}$$

so that

$$D_k(\bar{\Phi}\gamma^k\psi) = \bar{\Phi}(\overleftarrow{D}_k\gamma^k + \gamma^k\overrightarrow{D}_k)\psi, \tag{A.2}$$

where the derivatives on the right-hand side are spinorial covariant derivatives, defined by (2.16)–(2.18). Since the tetrad components γ^μ are constant, the left-hand side of (A.2) gives

$$D_k(\bar{\Phi}\gamma^k\psi) = \bar{\Phi}(\overleftarrow{\partial}_k\gamma^k + \gamma^k\overrightarrow{\partial}_k)\psi + \bar{\Phi}\gamma^\mu\lambda^k_{\mu|k}\psi. \tag{A.3}$$

The right-hand side of (A.2) is

$$\bar{\Phi}(\overleftarrow{D}_k\gamma^k + \gamma^k\overrightarrow{D}_k)\psi = \bar{\Phi}(\overleftarrow{\partial}_k\gamma^k + \gamma^k\overrightarrow{\partial}_k)\psi - \bar{\Phi}(\Gamma_k\gamma^k - \gamma^k\Gamma_k)\psi. \tag{A.4}$$

From the definition (2.16) of the spinorial connection Γ_k we have

$$\Gamma_k\gamma^k - \gamma^k\Gamma_k = -\tfrac{1}{4}\lambda^k_\mu\lambda^m_\alpha\lambda_{\beta m|k}(\sigma^{\alpha\beta}\gamma^\mu - \gamma^\mu\sigma^{\alpha\beta}). \tag{A.5}$$

Using the commutation relation

$$\sigma^{\alpha\beta}\gamma^\mu - \gamma^\mu\sigma^{\alpha\beta} = -2(\gamma^\alpha\eta^{\beta\mu} - \gamma^\beta\eta^{\alpha\mu}), \tag{A.6}$$

we find (A.5) becomes

$$\Gamma_k\gamma^k-\gamma^k\Gamma_k = \gamma^\alpha\lambda^k_\mu\lambda^m_\alpha\lambda^\mu_{m|k} = \gamma^m\lambda^k_\mu\lambda^\mu_{m|k} = -\gamma^m\lambda^k_{\mu|k}\lambda^\mu_m = -\gamma^\mu\lambda^k_{\mu|k}. \tag{A.7}$$

Substituting in (A.4) we find

$$\bar{\phi}(\overleftarrow{D}_k\gamma^k+\gamma^k\overrightarrow{D}_k)\psi = \bar{\phi}(\overleftarrow{\partial}_k\gamma^k+\gamma^k\overrightarrow{\partial}_k)\psi+\bar{\phi}\gamma^\mu\lambda^k_{\mu|k}\psi = D_k(\bar{\phi}\gamma^k\psi), \tag{A.8}$$

which is what we set out to prove.

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